Welcome to your Mathematics Bridging Task!

This Bridging Task is designed to consolidate your GCSE maths knowledge in the topic areas which form a foundation to the A-Level maths course. Without this knowledge you will not be able to access the A-Level maths content. It is therefore crucial that you have completed and fully understand everything from the summer task by September.

The Bridging Task is to work through the **'Head start to A-Level maths'** workbook, which can be found as a separate pdf document **'Head start to A level Maths**'. You must complete every question from the **Diagnostic test (Questions 1 – 43)** and bring it to your first maths lesson for inspection. If you struggle with any of the questions you must go straight to the relevant page in the textbook and brush up your skills. The rest of the workbook will provide useful guidance and additional practice to give you the best possible start to A-Level maths. There are '**Extension Problem-Solving Questions**' at the end of **<u>this</u>** document, everyone is welcome to have a go at these questions. However, if you are considering studying a Maths-based degree at University, you must attempt **at least Question 1, 2(1), 2(2), 2(3), 2(4) 3, 4(a-h)** of the Extension Questions.

All the answers to the 'Head start to A level Maths' at the back of the pdf textbook. You must clearly mark all your work and do corrections in a different colour pen. You must be able to fully understand how to answer every question from the diagnostic tests. Please be prepared to sit a short test on what you have covered in this Summer Task in September. The solutions to the Extension Tasks are the back of this document.

<u>How do we expect your work to be presented?</u>

You will need to get a notebook to do this summer task in. You will bring this notebook to your first maths lesson for your teacher to check your work. This notebook will continue to be used throughout year 12 for the independent study we set on a weekly basis.

<u>Which Calculator must you have for this course?</u>

The A-Level Maths exam <u>requires</u> all students to use a calculator for **all** papers.

It may be the calculator you used at GCSE <u>does not</u> meet the new criteria.

We would like you all to get a **<u>Casio fx-991CW</u>** calculator**.** Please check the code very carefully, there are similar ones available, but they may not be suitable please do not buy the Casio fx-85GT as this looks the same but has extremely limited functionality for A-Level Maths. PLEASE CHECK THE CODE!!!

- If you currently own the **Casio fx-991EX** this is the only viable alternative for an A-Level Maths calculator (so can be used) however Casio are not restocking or making this model anymore.
- **PLEASE DO NOT BUY or ORDER ANY FX-991EX models from the internet** as there are scammers at large selling fake/counterfeit ones.

Extension Problem-Solving Questions

Question 1 – Two-Way Algebra

Each column and row has a property which some equation or inequality may or may not have. If an example has the properties of the corresponding row and column, then it can appear in that cell.

We have omitted some headings, and some entries in cells. Can you complete the table?

		Solutions lie between 1 and 6		All values of x are solutions
	$x^2+4=0$	$5x-3=9-x$		
Inequalities			$8-x<2-3x$	
	$y=3x$ and $\frac{y}{3}-1=x$		$y=3x+5$ and $y=-x-3$	

- Did you have any choice about the row and column headings?
- Can you simplify any of your examples?
- Do all your examples require some 'solving' to check they fit the attributes of the cell? If not, can you make it so that they all do?
- If we required all the cells to contain quadratics, would it still be possible to fill all the cells?

Question 2 – Index Issues

$\sqrt{x}$ or $x^{\frac{1}{2}}$? Whichever we prefer, we need to be comfortable working with indices and roots in either form.

Below is a series of equations, for which you should think about the following questions.

> * For what values of x and y do these statements make sense?
>
> What are the possible values for the expressions?

(1) $(xy)^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}}$

(2) $(xy)^{\frac{5}{3}} = x^{\frac{5}{3}}y^{\frac{5}{3}}$

(3) $(xy)^{\frac{2}{3}} = x^{\frac{2}{3}}y^{\frac{2}{3}}$

(4) $(xy)^{-\frac{1}{2}} = x^{-\frac{1}{2}}y^{-\frac{1}{2}}$

(5) $(xy)^{\frac{1}{2}} = x^{\frac{1}{3}}y^{\frac{2}{3}}$

(6) $(xy)^{-2} = x^2y^2$

(7) $\left(\frac{x}{y}\right)^{-\frac{1}{3}} = x^{-\frac{1}{3}}y^{\frac{1}{3}}$

(8) $\left(\frac{x}{y}-\frac{1}{y}\right)^{\frac{1}{2}} = y^{-\frac{1}{2}}(x-1)^{\frac{1}{2}}$

Question 3 – Sequences (Food for thought)

Find the largest integer that divides every term of the sequence $1^5-1,\ 2^5-2,\ 3^5-3,\ \ldots,\ n^5-n,\ \ldots$.

Can you generalise your findings?

Hints

We could find the first few terms of the sequence, and look to see which numbers divide all of them.

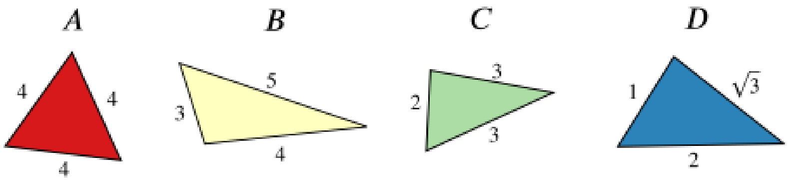
We could try a warm-up problem. What is the largest integer that divides every term of the sequence $1^3-1,\ 2^3-2,\ 3^3-3,\ \ldots,\ n^3-n$?

We can factorise $n^5-n=n(n^4-1)$. Can we factorise further by factorising n^4-1?

We could make a conjecture about what the answer is; that would be of the form "Every number of the form n^5-n is divisible by a" (with some suitable value of a). Then we might try proof by induction.

Question 4 – Similar Triangles

Here are some triangles.



The diagrams are not to scale.

Can you find ...

(a) ... a triangle similar to triangle A with perimeter 6?

(b) ... a triangle similar to triangle B with perimeter 6?

(c) ... a triangle similar to triangle C with perimeter 6?

(d) ... a triangle similar to triangle D with perimeter 6?

(e) ... a triangle similar to triangle A with area 10?

(f) ... a triangle similar to triangle B with area 10?

(g) ... a triangle similar to triangle C with area 10?

(h) ... a triangle similar to triangle D with area 10?

(i) ... a right-angled triangle similar to one of triangles A, B, C, D with area 12?

(j) ... an isosceles triangle similar to one of triangles A, B, C, D with area 12?

(k) ... an equilateral triangle similar to one of triangles A, B, C, D with area 12?

(l) ... a scalene triangle similar to one of triangles A, B, C, D with area 12?

(m) ... a right-angled triangle with area 12 that is not similar to any of triangles A, B, C, D?

(n) ... an isosceles triangle with area 12 that is not similar to any of triangles A, B, C, D?

(o) ... an equilateral triangle with area 12 that is not similar to any of triangles A, B, C, D?

(p) ... a scalene triangle with area 12 that is not similar to any of triangles A, B, C, D?

Recommended Books, Articles and Activities

How To Solve It: A New Aspect of Mathematical Method by George Polya - How to Solve It: A New Aspect of Mathematical Method: Amazon.co.uk: Polya, George: 9780140124996: Books

An introduction to Differentiation - An Introduction to Differentiation (maths.org)

Guided and Interactive Problem Solving activities (US) - Brilliant | Learn interactively

E16plus Newsletter - e16plus Newsletter - IMA

Fyfe, M. T., Jobbings, A. and Kilday, K. (2007) Progress to Higher Mathematics, Arbelos. ISBN 9780955547706.

Alternatively, also published by Arbelos is an expansion of the same book which is more specifically aimed at the transition to A Level (Year 13).

Fyfe, M. T., and Kilday, K. (2011) Progress to Advanced Mathematics, Arbelos. ISBN 9780955547737.

Possible Solutions to Extension Tasks

Question 1

	No real solutions	Solutions lie between 1 and 6	Negative solutions only	All values of x are solutions
Equations in x	$x^2+4=0$	$5x-3=9-x$	$2x-5=-12$	$(x-1)^2-1=x^2-2x$
Inequalities	$x^2<0$	$5<2x-3<15$	$8-x<2-3x$	$x^2\geq 0$
Simultaneous equations in x and y	$y=3x$ and $\frac{y}{3}-1=x$	$3y=x$ and $y=x-2$	$y=3x+5$ and $y=-x-3$	$y=2x$ and $\frac{y}{2}=x$

Question 2

(1) $(xy)^{\frac{1}{2}}=x^{\frac{1}{2}}y^{\frac{1}{2}}$

Is the left-hand side the same as the right-hand side?

$$(xy)^{\frac{1}{2}}=\sqrt{xy}=\sqrt{x}\sqrt{y}=x^{\frac{1}{2}}y^{\frac{1}{2}}$$

It would seem so from the reasoning above, however what happens if $x=-2$ and $y=-8$? $\sqrt{xy}$ will give us an answer of 4 but $\sqrt{x}\sqrt{y}$ will not be defined.

We can only get a real value from a square root when the number is not negative, so $x\geq 0$ and $y\geq 0$.

The overall value of the expression $(xy)^{\frac{1}{2}}$ (and therefore $x^{\frac{1}{2}}y^{\frac{1}{2}}$) will also be ≥ 0.

(2) $(xy)^{\frac{5}{3}}=x^{\frac{5}{3}}y^{\frac{5}{3}}$

$$(xy)^{\frac{5}{3}}=\left(\sqrt[3]{xy}\right)^5=\left(\sqrt[3]{x}\right)^5\left(\sqrt[3]{y}\right)^5=x^{\frac{5}{3}}y^{\frac{5}{3}}$$

Is there anywhere this chain of reasoning breaks down? Does it make a difference if we write $\sqrt[3]{x^5}$ instead of $\left(\sqrt[3]{x}\right)^5$?

We can cube root any real value, positive or negative, so $x,y\in\mathbb{R}$. What values can $(xy)^{\frac{5}{3}}$ take?

(3) $(xy)^{\frac{2}{3}}=x^{\frac{2}{3}}y^{\frac{2}{3}}$

$$(xy)^{\frac{2}{3}}=\left(\sqrt[3]{xy}\right)^2=\left(\sqrt[3]{x}\right)^2\left(\sqrt[3]{y}\right)^2=x^{\frac{2}{3}}y^{\frac{2}{3}}$$

We know we can cube root any number, so $x,y\in\mathbb{R}$. The value of $(xy)^{\frac{2}{3}}$ however, is restricted. How is it different from the value of $(xy)^{\frac{5}{3}}$ and why is that?

(4) $(xy)^{-\frac{1}{2}}=x^{-\frac{1}{2}}y^{-\frac{1}{2}}$

This is similar to the first question. It only involves square roots, so we know x and y cannot be negative. What happens to either side of the equation if x or $y=0$?

(5) $(xy)^{\frac{1}{2}} = x^{\frac{1}{3}} y^{\frac{2}{3}}$

On first glance, we might easily dismiss this as being incorrect. The question that was asked though, is 'for what values of x and y do these statements make sense?' Are there any values of x and y when this equation would be true?

If $x = 0$ or $y = 0$, then the equation is satisfied, but to see if there are any other possibilities we should try to solve this equation.

$$(xy)^{\frac{1}{2}} = x^{\frac{1}{3}} y^{\frac{2}{3}}$$

For the left hand side to be defined, $xy \geq 0$. In addition, x and y cannot be negative. Can you explain why?

We can write

$$x^{\frac{1}{2}} y^{\frac{1}{2}} = x^{\frac{1}{3}} y^{\frac{2}{3}},$$

then divide by $x^{\frac{1}{3}} y^{\frac{1}{2}}$, assuming $x, y \neq 0$,to get

$$x^{\frac{1}{6}} = y^{\frac{1}{6}}.$$

Therefore

$$x = y$$

is a solution for $x, y > 0$, along with $x = 0$ or $y = 0$.

(6) $(xy)^{-2} = x^2 y^2$

Once again, this is not true for all values, but may still be true for certain values of x and y.

It can also be written

$$\frac{1}{(xy)^2} = x^2 y^2,$$

so we can see that if x or y was zero the left-hand side would not exist, therefore $x, y \neq 0$.

We can multiply by $(xy)^2$

$$1 = x^4 y^4,$$

then take the fourth root. This gives us a positive and negative value so we get

$$1 = xy \text{ or } -1 = xy,$$

giving us the solutions

$$x = \frac{1}{y} \text{ and } x = -\frac{1}{y}.$$

(7) $\left(\frac{x}{y}\right)^{-\frac{1}{3}} = x^{-\frac{1}{3}} y^{\frac{1}{3}}$

From the $\left(\frac{x}{y}\right)$ on the left hand side we can see that $y \neq 0$, even though it could be on the right hand side.

On the right hand side we have $x^{-\frac{1}{3}}$ which is $\frac{1}{\sqrt[3]{x}}$ and so $x \neq 0$.

Otherwise we can cube root any value so $x, y \in \mathbb{R}$ except for $x, y \neq 0$ and the value of the expressions can also take any value except 0.

(8) $\left(\frac{x}{y}-\frac{1}{y}\right)^{\frac{1}{2}}=y^{-\frac{1}{2}}(x-1)^{\frac{1}{2}}$

On the left hand side we require $y \neq 0$ and $\frac{x}{y}-\frac{1}{y} \geq 0$, which implies that $x \geq 1$or $x \leq 1$ depending on whether y is positive or negative.

On the right hand side we have $y^{-\frac{1}{2}}$ which requires $y>0$, and $(x-1)^{\frac{1}{2}}$ which also requires $x \geq 1$.

Putting these together gives us $y>0$ and $x \geq 1$ and the value of either side of the equation must be ≥ 0.

Question 3

The general term of this sequence is n^5-n. We can factorise this expression as follows, using the expression for the difference of two squares:

$$\begin{aligned} n^5-n &= n(n^4-1) \\ &= n(n^2-1)(n^2+1) \\ &= n(n-1)(n+1)(n^2+1). \end{aligned}$$

This expression is divisible by 2. That's because n and $n+1$ are two consecutive integers, so one of them must be even and the other odd. Then, as n^5-n is divisible by both n and $n+1$, it has at least one even factor and must therefore be even (the product of an even integer and any other integer is always even).

This expression is also divisible by 3. That's because $n-1$, n and $n+1$ are three consecutive integers, so one of them must be a multiple of 3. We can see this by considering the remainder left upon dividing n by 3: the only possible values are 0, 1, and 2. If it is 0, then n is a multiple of 3. If it is 3, then $n-1$ is a multiple of 3. If it is 2, then $n+1$ is a multiple of 3.

Similarly to above, since n^5-n is divisible by $n-1$, n, and $n+1$, it must have a factor which is a multiple of 3, and therefore must itself be divisible by 3.

This expression is also divisible by 5, although this is slightly tricker to show than in the previous two parts. Firstly, we consider the remainder left when we divide n by 5. This can take the values 0, 1, 2, 3, and 4. We'll consider the five cases separately.

(1) If this remainder is 0, then n itself is divisible by 5, and then so is n^5-n, since it is divisible by n.

(2) If this remainder is 1, then $n-1$ is divisible by 5, and then so is n^5-n, as it is divisible by $n-1$.

(3) If this remainder is 2, then n is 2 greater than a multiple of 5. That is, we can write $n=5k+2$ for some integer k. Then

$$\begin{aligned} n^2 + 1 &= (5k+2)^2 + 1 \\ &= 25k^2 + 20k + 4 + 1 \\ &= 25k^2 + 20k + 5 \\ &= 5(5k^2 + 4k + 1). \end{aligned}$$

As k is an integer, $5k^2 + 4k + 1$ is also an integer, and so $n^2 + 1$ is a multiple of 5. Then so is $n^5 - n$, as it is divisible by $n^2 + 1$.

(4) Similarly, if this remainder is 3, then we can write $n = 5m + 3$, for some integer m. Then

$$\begin{aligned} n^2 + 1 &= (5m+3)^2 + 1 \\ &= 25m^2 + 30m + 10 \\ &= 5(5m^2 + 6m + 2). \end{aligned}$$

So again, $n^2 + 1$ is a multiple of 5, meaning that $n^5 - n$ is too.

(5) If the remainder is 4, then $n + 1$ is divisible by 5, and then so is $n^5 - n$, as it is divisible by $n + 1$.

We have shown that, for all n, $n^5 - n$ is divisible by 2, 3, and 5. This means that every term in the sequence is divisible by the lowest common multiple of 2, 3 and 5. In this case this is simply their product, 30, as they have no common prime factors.

So 30 divides every number in the sequence. This means that the largest integer which divides every term in the sequence must be at least 30.

Now, look at the second term in the sequence: $2^5 - 2$. This is equal to 30, which obviously is not divisible by any integers greater than itself. So, 30 is the largest integer which divides every term in the sequence.

Question 4

(a) ... a triangle similar to triangle A with perimeter 6?

Triangle A has perimeter $4 + 4 + 4 = 12$, so we want to scale by factor $6/12 = 1/2$, so we should choose the triangle with side lengths 2, 2 and 2.

(b) ... a triangle similar to triangle B with perimeter 6?

Triangle B has perimeter $3 + 4 + 5 = 12$ (like triangle A), so we again use factor $1/2$, so we should choose the triangle with side lengths $3/2$, 2 and $5/2$.

(c) ... a triangle similar to triangle C with perimeter 6?

Triangle C has perimeter $3 + 3 + 2 = 8$, so we want to scale by factor $6/8 = 3/4$. So we should choose the triangle with side lengths $9/4$, $9/4$ and $3/2$.

(d) ... a triangle similar to triangle D with perimeter 6?

Triangle D has perimeter $1+2+\sqrt{3}=3+\sqrt{3}$, so we want to scale by factor

$$\frac{6}{3+\sqrt{3}}=\frac{6(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}=3-\sqrt{3}.$$

So we should choose the triangle with side lengths $3-\sqrt{3}$, $6-2\sqrt{3}$ and $-3+3\sqrt{3}$.

(e) ... a triangle similar to triangle A with area 10?

We need to remember that when we scale the lengths of a triangle by a certain factor, we scale the area of the triangle by the *square* of that factor. So we must allow for that.

Triangle A has area $4\sqrt{3}$, so we want to scale by factor

$$\sqrt{\frac{10}{4\sqrt{3}}}=\sqrt{\frac{5\sqrt{3}}{6}}.$$

So we should choose the triangle with side lengths $4\sqrt{\frac{5\sqrt{3}}{6}}$, $4\sqrt{\frac{5\sqrt{3}}{6}}$ and $4\sqrt{\frac{5\sqrt{3}}{6}}$.

(f) ... a triangle similar to triangle B with area 10?

Triangle B has area 6, so we want to scale by factor

$$\sqrt{\frac{10}{6}}=\frac{\sqrt{15}}{3}.$$

So we should choose the triangle with side lengths $\sqrt{15}$, $\frac{4\sqrt{15}}{3}$ and $\frac{5\sqrt{15}}{3}$.

(g) ... a triangle similar to triangle C with area 10?

Triangle C has area $2\sqrt{2}$, so we want to scale by factor

$$\sqrt{\frac{10}{2\sqrt{2}}}=\sqrt{\frac{5\sqrt{2}}{2}}.$$

So we should choose the triangle with side lengths $\sqrt{10\sqrt{2}}$, $3\sqrt{\frac{5\sqrt{2}}{2}}$, $3\sqrt{\frac{5\sqrt{2}}{2}}$.

(h) ... a triangle similar to triangle D with area 10?

Triangle D has area $\frac{\sqrt{3}}{2}$, so we want to scale by factor

$$\sqrt{\frac{10}{\sqrt{3}/2}}=\sqrt{\frac{20}{\sqrt{3}}}=\frac{2\sqrt{5}}{\sqrt[4]{3}}.$$

So we should choose the triangle with side lengths $\frac{2\sqrt{5}}{\sqrt[4]{3}}$, $\frac{4\sqrt{5}}{\sqrt[4]{3}}$ and $\frac{2\sqrt{15}}{\sqrt[4]{3}}$.

(i) ... a right-angled triangle similar to one of triangles A, B, C, D with area 12?

Of the four given triangles, only triangles B and D are right-angled. We can check that using Pythagoras's theorem, since the side lengths 3, 4 and 5 satisfy his equation, as do the side lengths 1, $\sqrt{3}$ and 2.

Let's find a triangle similar to B with area 12. And we can do that using the same approach as earlier.

Triangle B has area 6, so we want to scale by factor

$$\sqrt{\frac{12}{6}} = \sqrt{2}.$$

So we should choose the triangle with side lengths $3\sqrt{2}$, $4\sqrt{2}$ and $5\sqrt{2}$.

(j) ... an isosceles triangle similar to one of triangles A, B, C, D with area 12?

Of the four given triangles, only triangles A and C are isosceles. We could choose either, but let's choose C and save A for later since it's equilateral too.

Triangle C has area $2\sqrt{2}$, so we want to scale by factor

$$\sqrt{\frac{12}{2\sqrt{2}}} = \sqrt{3\sqrt{2}}.$$

So we should choose the triangle with side lengths $2\sqrt{3\sqrt{2}}$, $3\sqrt{3\sqrt{2}}$ and $3\sqrt{3\sqrt{2}}$.

(k) ... an equilateral triangle similar to one of triangles A, B, C, D with area 12?

Our only option here is triangle A.

Triangle A has area $4\sqrt{3}$, so we want to scale by factor

$$\sqrt{\frac{12}{4\sqrt{3}}} = \sqrt[4]{3}.$$

So we should choose the triangle with side lengths $4\sqrt[4]{3}$, $4\sqrt[4]{3}$ and $4\sqrt[4]{3}$.

(l) ... a scalene triangle similar to one of triangles A, B, C, D with area 12?

Here we can choose between triangles B and D. Since we've already done B, let's pick D.

Triangle D has area $\frac{\sqrt{3}}{2}$, so we want to scale by factor

$$\sqrt{\frac{12}{\sqrt{3}/2}} = \sqrt{8\sqrt{3}} = 2\sqrt{2\sqrt{3}}.$$

So we should choose the triangle with side lengths $2\sqrt{2\sqrt{3}}$, $4\sqrt{2\sqrt{3}}$ and $2\sqrt{6\sqrt{3}}$.

(m) … a right-angled triangle with area 12 that is not similar to any of triangles A, B, C, D?

We have lots of choice here, we just have to make sure that our right-angled triangle isn't similar to triangle B or D(it definitely won't be similar to either of the others).

Let's try to be lazy when constructing an example. An easy way to make a triangle have area 12 is to give it base 12 and height 2, for example, and then we can use Pythagoras to finish off.

Choose the triangle with side lengths 2, 12 and $\sqrt{2^2+12^2}=\sqrt{4+144}=\sqrt{148}=2\sqrt{37}$. The side lengths satisfy Pythagoras's equation so the triangle is right-angled, and its area is 12. It is easy to check that it is not similar to triangle B or triangle D, by looking at ratios of two sides (e.g. $\frac{3}{4}\neq\frac{2}{12}$).

(n) … an isosceles triangle with area 12 that is not similar to any of triangles A, B, C, D?

We need to avoid both A and C here. It should be quite straightforward to build an isosceles triangle with area 12, by picking the base and height and then finding the missing side length.

Choose the isosceles triangle with base 2 (as one side of the triangle) and height 12. The other sides have length $\sqrt{1^2+12^2}=\sqrt{145}$. This is clearly isosceles, clearly has area 12, and cannot be similar to triangle C (since they have the same base but different heights).

(o) … an equilateral triangle with area 12 that is not similar to any of triangles A, B, C, D?

This is a bit tricky…

Every equilateral triangle is similar to every other equilateral triangle! So any equilateral triangle will be similar to A, so there is no triangle that meets the criteria.

(p) … a scalene triangle with area 12 that is not similar to any of triangles A, B, C, D?

There are lots and lots of examples here. Again, a nice approach might be to specify the base and height, since they determine the area, and then just to make sure that the triangle is not isosceles or right-angled. The hardest part might be calculating the side lengths!